4. Mandel J., Sur les ondes de choc plastiques en milieu indéfini isotrope. C.R. Akad. Sci. Paris, 253, 1961.
5. Kulikovskii A.G., Surfaces of discontinuity separating two perfect media of different properties. Recombination waves in magnetohydrodynamics. PMM Vol. 32, №6, 1968.
6. Freudenthal A.M. and Geiringer H., The Mathematical Theories of the Inelastic Continuum, Handbuch der Physik, S. Flugge ed., Vol. 6, Springer-Verlag, 1958.
7. Bleich H.H. and Nelson I., Plane waves in an elastic-plastic half space due to combined surface pressure and shear. Trans. ASME, J. of Applied Mechanics, Series E, Vol. 33, №1, 1966.

Translated by L.K.

UDC 517.946

ASYMPTOTIC METHODS OF SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

PMM Vol. 36, №2, 1972, pp. 330-343
M.I. RABINOVICH and A.A. ROZENBLIUM
(Gor'kii)
(Received May 3, 1971)

The asymptotic method presented here for one-dimensional nonlinear dynamic systems described in terms of partial differential equations with a small parameter, uses a known solution of the unperturbed problem as the basis for constructing an approximate solution on the prescribed variable range, which will tend to its exact value when the small parameter tends to zero. The method is based essentially on varying the arbitrary constants entering the unperturbed solution and constructing, for the slowly varying functions of the coordinate and time thus created, a system of differential equations the form of which depends on the degree of approximation. These equations remain nonlinear in the partial derivatives thus retaining the specific character of the problem and are, at the same time, easier to analyze than the initial equations.

The substantiation of the method is reduced to proving a theorem on continuous dependence of the solution of the system of partial differential equations on the variation of its right-hand sides, and the proof is given here for hyperbolic and symmetrical parabolic systems.

The procedure considered here embraces, as its particular cases, the known asymptotic methods of the perturbation theory [1, 2] of the geometrical optics [3, 4] and the methods [5, 6] related to the method for ordinary differential equations which are almost linear [7].

1. Let us consider a system of differential equations of the form

$$N(u) \equiv u_t + A(u, x, t, \chi, \tau) u_x + B(u, x, t, \chi, \tau) =$$

$$= \sum_{k=1}^{\infty} \mu^k f_k (u, u_x, u_t, x, t, \chi, \tau) \tag{1.1}$$

$$(\chi = \chi^\circ + \mu x, \ \tau = \tau^\circ + \mu t, \ 0 < \mu \ll 1)$$

with the initial data

$$u(x, t, \mu)|_\Gamma = \Phi(x, t, \mu) \tag{1.2}$$

where $u = (u_1, \dots u_n)$ is an unknown vector function, A is a square matrix, B and f_k are n-dimensional vector functions, A, B and f_k being as smooth the functions of their arguments as will be later required, x is the spatial coordinate, t is time and Γ denotes the initial value curve (*) Let us also assume known the family of solutions of the unperturbed problem, i.e. of the system (1.1), (1.2) with $\mu = 0$ dependent on $l + r$ parameters

$$V(x, t, \chi^\circ, \tau^\circ) = V[C^{(1)}(x, t, \chi^\circ, \tau^\circ), C^{(2)}(\chi^\circ, \tau^\circ)] \tag{1.3}$$

$$(N\{V(x, t, \chi^\circ, \tau^\circ)\}|_{\mu=0} = 0, \quad C^{(1)} = [C_1(x, t, \chi^\circ, \tau^\circ), \dots$$

$$\dots, C_l(x, t, \chi^\circ, \tau^\circ)], \quad C^{(2)} = [C_{l+1}, \dots, C_{l+r}])$$

Here $C^{(1)}$ denotes the generalized phases, x and t are known functions and $C^{(2)}$ denotes arbitrary constants.

The aim of the proposed asymptotic method is to construct, on the basis of a known solution V of the unperturbed problem, an approximate solution $u^{(m)}(x, t)$ of the system (1.1), (1.2) satisfying on the finite intervals X and T the following condition:

$$|u^{(m)}(x, t, \mu) - u(x, t, \mu)| < M\mu^{(m)} \quad (M = M(X, T)) \tag{1.4}$$

where $u(x, t, \mu)$ is the exact solution of (1.1), (1.2).

Following the usual procedure adopted in constructing the asymptotic methods (see [7]) we separate the problem into two parts. First we obtain the function $u^{(m)}$ satisfying the system (1.1), (1.2) with accuracy up to the terms of order μ^m This in fact is the essence of the method, given in Sect. 2. Next we prove that $u^{(m)}$ satisfies (1.4) thus providing the substantiation of the method (Sect. 3).

The method essentially consists of varying the arbitrary constants entering the unperturbed solution (1.3) and constructing a system of partial differential equations for the resulting functions of the coordinate and time. The method is not subject to any limitations that may have been imposed on the initial system (1.1), the only requirement being that a sufficiently general family of solutions (1.3) exists when $\mu = 0$. For this reason, all conditions which must be fulfilled by the system of equations considered in order to satisfy the inequality (1.4), may be formulated at the second stage of solution, namely when substantiating the method.

Varying the generalized phases and arbitrary constants of the solution (1.3) in χ and τ we construct the following m th approximation for the system (1.1), (1.2)

$$u^{(m)} = V[\bar{C}^{(1)}(x, t, \chi, \tau), \bar{C}^{(2)}(\chi, \tau)] + \sum_{i=1}^{m} \mu^i w^{(i)}(x, t, \chi, \tau) \tag{1.5}$$

---

*) Such a statement of the problem with the initial data includes the Cauchy and the boundary value problems as particular cases.

where the functions $w^{(i)}$ to be determined as well as V both satisfy the initial values

$$|V-u|_\Gamma < K^{(0)}\mu, \quad \left|V+\sum_{i=1}^{j}\mu^i w^{(i)}-u\right|_\Gamma < K^{(j)}\mu^{j+1} \quad (j=1,2,\ldots,m) \tag{1.6}$$

following from (1.2) and (1.5). The reintroduced function $V(x, t, \chi, \tau)$ (with $\mu \neq 0$) can be determined by the system

$$\frac{\partial^* V}{\partial t} = \frac{\partial V}{\partial t} + \sum_{i=1}^{m}\mu^i F^{(i)}[\chi, \tau, V_{c_1}, \ldots, V_{c_{l+r}}] \qquad \left(\frac{\partial^*}{\partial t} = \frac{\partial}{\partial t} + \mu\frac{\partial}{\partial \tau}\right) \tag{1.7}$$

where $F^{(i)}$ are, for the time being, unknown operators in χ and τ [1](*) and the initial values for V are given by (1.6). In determining $F^{(i)}$ and $w^{(i)}$ we shall use the condition that $u^{(m)}$ must satisfy the system (1.1), (1.2) with the accuracy up to the terms of order μ^{m+1}.

Let us consider the operator

$$\dot{N}(u^{(m)}) = N(u^{(m)}) - \sum_{k=1}^{\infty}\mu^k f_k$$

and arrange, using the Taylor expansion, its terms in increasing powers of μ from 1 to m. Taking at the same time into account (1.5) and (1.7), we obtain

$$\begin{aligned}
\dot{N}(u^{(m)}) &= \mu\,[Lw^{(1)} + F^{(1)} + A(V)V_\chi - f_1(V, V_x, V_t)]_{(1)} + \\
&+ \mu^2\Big[Lw^{(2)} + F^{(2)} + w_\tau^{(1)} + A(V)w_\chi^{(1)} + A_u'(V)w_x^{(1)}w^{(1)} + \frac{1}{2}A_u''(V)(w^{(1)})^2 + \\
&+ f_{1_u}(V, V_x, V_t)w^{(1)} - f_{1_{u_x}}(V, V_x, V_t)(w_x^{(1)} + V_\chi) - f_{1_{u_t}}(V, V_x, V_t)\times \\
&\times (w_t^{(1)} + V_\tau) + f_2(V, V_x, V_t)\Big]_{(2)} + \ldots + \mu^m\Big[Lw^{(m)} + F^{(m)} + w_\tau^{(m-1)} + \\
&+ A(V)w_\chi^{(m-1)} + A_u(V)(w^{(m-1)}w_x^{(1)} + \ldots + w^{(1)}w_x^{(m-1)} + \frac{1}{2}A_u''(V)(w^{(1)})^2\times \\
&\times w_x^{(m-2)} + \ldots) + \ldots + \frac{1}{(m-1)!}A_u^{m-1}(V)((w^{(1)})^{m-1}w_x^{(1)} - f_{1_u}(V, V_x, V_t)\times \\
&\times w^{(m-1)} - \ldots - f'_{m-1}(V, V_x, V_t)w^{(1)} - f_m(V, V_x, V_t)\Big]_{(m)} + \\
&+ \mu^{m+1}[w_\tau^{(m)} + A(V)w_\chi^{(m)} + \ldots]_{(m+1)}
\end{aligned} \tag{1.8}$$

where the following notation is used

$$Lw^{(i)} = w_t^{(i)} + A(V)w_x^{(i)} + Pw^{(i)}, \qquad P = B_u + A_u V_x \tag{1.9}$$

Setting the coefficients of μ^i ($i = 1, 2, \ldots, m$) equal to zero we obtain from (1.8) the following linear system of equations for $w^{(i)}$

$$Lw^{(i)} = h^{(i)}(x, t, \chi, \tau) - F^{(i)} \quad (i=1,2,\ldots,m), \qquad h^{(i)} = [\ \ ]_{(i)} - Lw^{(i)} - F^{(i)} \tag{1.10}$$

where $h^{(i)}$ depends on $w^{(k)}$, $w_\chi^{(k)}$, $w_\tau^{(k)}$, $w_x^{(k)}$ and $w_t^{(k)}$ ($k = 1, 2, \ldots, i-1$) and the operator L is taken over the fast variables x and t. For this reason, when $w^{(i)}$ in (1.10) are determined successively, χ and τ serve as parameters.

If $w^{(i)}$ and their derivatives determined by the problem (1.10), (1.6) are of order V

$$w_t^{(i)}, w_x^{(i)}, w_\chi^{(i)}, w_\tau^{(i)}, w^{(i)} \sim V \tag{1.11}$$

*) Such an approach for the case of partial differential equations was first employed in [5]

then the function $g^{(m)}(x, t, \chi, \tau, \mu) = [\ ]_{(m+1)}$ is of the same order and by virtue of the choice of $w^{(i)}$ the mth approximation $u^{(m)}$ satisfies the initial system (1.1), (1.2) with the accuracy up to the terms of order μ^{m+1}, i.e. it satisfies the system

$$N(u^{(m)}) = \sum_{k=1}^{m} \mu^k f_k + \mu^{m+1} g^{(m)} \tag{1.12}$$

The problem of determining the restricted (in the sense of (1.11)) functions from (1.10) and (1.6) can be solved in the general case for certain right-hand sides of (1.10). The conditions which must be imposed on the right-hand sides of (1.10) to make the system solvable, are the initial conditions for determining the operators $F^{(i)}$. If this problem has a solution for any $h^{(i)}$, then we set $F^{(i)} \equiv 0$ and our procedure becomes then identical with the perturbation method. The manner of determining $F^{(i)}$ depends on the properties of the operator N and on the form of V.

2. Using what now follows as an important and a very general example, we give a method of obtaining $F^{(i)}$ for the case when the system (1.10), (1.6) admits the existence of solutions periodic in x and t. (*) Let $\Lambda(\chi, \tau)$ and $\Theta(\chi, \tau)$ denote the periods of these solutions with respect to x and t ($\Lambda \ll X$, $\Theta \ll T$). Let us formulate for $w^{(i)}$ a problem with periodic boundary conditions

$$w^{(i)}(x+\Lambda, t, \chi, \tau) = w^{(i)}(x, t+\Theta, \chi, \tau) = w^{(i)}(x, t, \chi, \tau) \tag{2.1}$$

Now, in addition to the boundary value problem (1.10), (2.1), let us also consider a homogeneous boundary value problem with a conjugate differential expression

$$L^*\psi \equiv -\psi_t - (A^*\psi)_x + P^*\psi = 0 \tag{2.2}$$

and boundary conditions which are also periodic

$$\psi(x+\Lambda, t, \chi, \tau) = \psi(x, t+\Theta, \chi, \tau) = \psi(x, t, \chi, \tau) \tag{2.3}$$

where A and P are matrices, Λ- and Θ-periodic in x and t, while A^* and P^* are matrices which transpose into A and P.

We shall show that the operator L^* of the boundary value problem (2.2), (2.3) is a conjugate of the operator L of the boundary value problem (1.10), (2.1). Scalar multiplying Lw by ψ and $L^*\psi$ by w we can write

$$(\psi, Lw) - (L^*\psi, w) = (\psi, w)_t + \frac{\partial}{\partial x}(\psi, Aw) \tag{2.4}$$

(since $(\psi, Bw) = (B^*\psi, w)$ and $(\psi, Aw) = (A^*\psi, w)$). Integration over the periods now yields

$$\int_x^{x+\Lambda}\int_t^{t+\Theta} \{(\psi, Lw) - (L^*\psi, w)\}\,dx\,dt = \int_x^{x+\Lambda} (\psi[x, t+\Theta], w[x, t+\Theta])\,dx -$$

$$\int_x^{x+\Lambda} (\psi[x,t], w[x,t])\,dx + \int_t^{t+\Theta} (\psi[x+\Lambda, t], A[x+\Lambda, t]\, w[x+\Lambda, t])\,dt -$$

*) We note that if (1.1) has the form $u_t + A(x, t)\,u_x = \mu f_1(x, t, u) + \ldots$ and the initial values are given on the characteristic, then the problem of determining $F^{(i)}$ becomes an algebraic one irrespective of the form of the solution V.

$$\int_t^{t+\theta} (\psi[x,t], A[x,t]\, w[x,t])\, dt = 0 \tag{2.5}$$

Thus the following equation holds for any w and ψ satisfying (2.1) and (2.3), respectively

$$\int_x^{x+\Lambda}\int_t^{t+\theta} (\psi, Lw)\, dxdt = \int_x^{x+\Lambda}\int_t^{t+\theta} (L^*\psi, w)\, dxdt \quad \text{or} \quad [\psi, Lw] = [L^*\psi, w] \tag{2.6}$$

and this equation defines the conjugate operator, i.e. L^* is a conjugate of L.

Since $L^*\psi = 0$ and $Lw = h - F$, from (2.6) we obtain

$$[L^*\psi,\ w] = [\psi,\ Lw] = [\psi,\ (h - F)] = 0 \tag{2.7}$$

Consequently, the necessary condition for a periodic solution of the problem (1.6), (1.10) (or a solution of the problem (1.1), (2.1)) to exist is, that $h^{(i)} - F^{(i)}$ is orthogonal (in the sense of the integral (2.6)) to any solution $\psi^{(j)}$ of the conjugate problem (2.2), (2.3), i.e. the operators must satisfy the following system of integral equations

$$\int_x^{x+\Lambda}\int_t^{t+\theta} (h^{(i)}, \psi^{(j)})\, dxdt = \int_x^{x+\Lambda}\int_t^{t+\theta} (F^{(i)}, \psi^{(j)})\, dxdt \tag{2.8}$$

or $[h^{(i)}, \psi^{(j)}] = [F^{(i)}, \psi^{(j)}]$.

To formulate the sufficient conditions of solvability of the problem (1.10), (2.1), we must define concretely the properties of the operator L. In particular, for the elliptic operator L it can be shown [8] that the conditions (2.8) are not only necessary, but also sufficient. They are also sufficient when the problem of determining $w^{(i)}$ by expanding $h^{(i)}$ and $w^{(i)}$ over the complete system of functions can be reduced to an algebraic problem [5, 6].

Comparing the present method with the perturbation method we must note that the presence in the right-hand side of (1.10) of the terms $F^{(i)}$ not previously determined, makes it possible to obtain restricted (in the sense of (1.11)) or periodic solutions for $w^{(i)}$ in the cases when the perturbation solution method cannot be used. Indeed, in the perturbation method equations analogous to (1.10) are also obtained for the functions $w^{(i)}$ but their right-hand sides, i.e. $h^{(i)}$, are orthogonal to all solutions of the corresponding conjugate system only in isolated cases. In the physical sense, such orthogonality excludes the possibility of resonances appearing in $h^{(i)}$ and this is precisely the condition of applicability of the perturbation method. In view of the fact that V, being a function of χ and τ, varies in the present method from one approximation to the next, it is possible to utilize the freedom of choice of $F^{(i)}$ to determine $V(x, t, \chi, \tau)$ in such a manner that the resonant terms of order μ^i are extracted from $h^{(i)}$

In the general case the conditions (2.8) lead to an infinite system of equations defining $F^{(i)}$ In order for this system to have a solution, the solution $V[\ ]$ must contain an infinity of the generalized phases and arbitrary constants. However in the real problems the number p of different functions $\psi^{(j)}$ for which $[h^{(i)}, \psi^{(j)}] = [F^{(i)}, \psi^{(j)}] \neq 0$ is obviously finite (this corresponds to a finite number of internal resonances in the system (1.10), (2.1)). Conditions (2.8) can therefore be satisfied if the number of the generalized phases and arbitrary constants $l + r \geqslant p$ (if $l + r > p$, then a part of these functions should be assumed predetermined from the initial data and therefore independent of χ and τ).

If $l + r < p$, the class of solutions of the unperturbed system must be enlarged so that $l' + r' = p$. When $l + r \geqslant p$, we can use (2.8) to write the operators $F^{(i)}$ out in full and finally obtain equations for the generalized phases $C_1, \ldots, C_l$ and parameters $C_{l+1} \ldots, C_{l+r}$ directly from (1.7).

Let us obtain these equations for the case when $C_1(x, t), \ldots, C_l(x, t)$ are linear functions of x and t, with $\mu = 0$. Most of the problems on propagation and interaction of waves in nonlinear media [9] are covered by this case.

First we shall show the validity of the relations

$$[V'_{c_i}, \psi^{(j)}] = \delta_{ij}\alpha_i \tag{2.9}$$

where α_i is a number.

Differentiating (1.4) with respect to C_i we obtain

$$L(V_{c_i}') = 0 \tag{2.10}$$

i.e. V_{c_i} are solutions of the homogeneous boundary value problem (1.10), (2.1).

Scalar multiplying (2.10) by ψ and (2.2) by $z = V_{c_i}$ and subtracting the results obtained, we find

$$(z, \psi)_t + (z, A^*\psi)_x = 0 \tag{2.11}$$

As the initial values for ψ can be chosen arbitrarily, we can assume without loss of generality that the orthogonality conditions hold for $t = 0$ (*)

$$\int\limits_x^{x+\Lambda} (z^{(i)}[x, 0], \psi^{(j)}[x, 0])\,dx = \delta_{ij}\frac{\alpha_i}{\theta} \tag{2.12}$$

Then, integrating (2.11) with respect to x and t from x to $x + \Lambda$ and from 0 to t, respectively, we obtain

$$\delta_{ij}\frac{\alpha_i}{\theta} = \int\limits_x^{x+\Lambda} (z^{(i)}[x, t], \psi^{(j)}[x, t])\,dx + \int\limits_0^t\int\limits_x^{x+\Lambda} (z^{(i)}[x, t], A^*[x, t]\,\psi^{(j)}[x, t])_x\,dx\,dt$$

from which, taking into account the periodic character of z, ψ and A, we finally obtain

$$\int\limits_x^{x+\Lambda}\int\limits_t^{t+\theta} (z^{(i)}, \psi^{(j)})\,dx\,dt = \delta_{ij}\alpha_i$$

Since C_i are linear in x and t when $\mu = 0$, we can assume that $\partial^* C_i/\partial t$ do not depend explicitly on x and t when $\mu \neq 0$ and solve Eqs. (1.7) for $\partial^* C_i / \partial t$. Scalar multiplying (1.7) in which

$$\frac{\partial^* V}{\partial t} = \frac{\partial V}{\partial t} + \mu\sum_{i=1}^{l+r} V'_{c_i}\frac{\partial C_i}{\partial \tau}$$

by $\psi^{(j)}$ we obtain, using (2.9), the equations for the parameters $C_{l+1}, \ldots, C_{l+r}$ and the generalized phases $C_1(x, t, \chi, \tau), \ldots, C_l(x, t, \chi, \tau)$ in the form

$$\alpha_i\frac{\partial C_i}{\partial \tau} = [h^{(1)}, \psi^{(i)}] + \ldots + \mu^{m-1}[h^{(m)}, \psi^{(i)}] \qquad (i = l+1, \ldots, l+r) \tag{2.13}$$

---

*) The initial conditions for $z^{(i)}$ can be used as the initial conditions for $\psi^{(j)}$ (provided they are orthogonalized) and this establishes a relation between the solutions of the initial and the conjugate problem.

$$\alpha_i \frac{\partial^* C_i}{\partial t} = [(-A(V)V_x' - B(V)), \psi^{(i)}] + \sum_{k=1}^{m} \mu^k [h^{(k)}, \psi^{(i)}] \qquad (i = 1, 2, \ldots, l) \tag{2.14}$$

Equations for the amplitudes and phases of the waves interacting in a weakly nonlinear medium [5, 6] follow from (2.13) and (2.14) as a particular case.

Relations (1.4) and (2.8) also yield equations for a group of methods analogous to the method of geometrical optics. To obtain these equations we expand C_i into a series in μ

$$C_i = \sum_{k=0}^{m} C_i^{(k)}(\chi, \tau)\mu^k \quad (i = l+1, \ldots, l+r), \qquad C_i = C_i^\circ(x, t, \chi, \tau) + \sum_{k=1}^{m} C_i^{(k)}(\chi, \tau)\mu^k \quad (i = 1, 2, \ldots, l)$$

and, instead of equations for V, we seek the equations for $C_i^{(k)}$. Moreover, $h^{(k)}$ in (1.10) should be replaced by $\bar{h}^{(k)}$, where $\bar{h}^{(k)} = h^{(k)}$ $(k = 2, 3, \ldots, m)$ and $\bar{h}^{(1)} = h^{(1)} - A(V)\,\partial V / \partial\chi$, and the following expression used for $F^{(k)}$:

$$F^{(k)} = \sum_{i=1}^{l+r} \left[V_{C_i} \frac{\partial C_i^{(k)}}{\partial \tau} + A(V) \frac{\partial C_i^{(k)}}{\partial x} V_{C_i}' \right] \qquad (k = 1, 2, \ldots, m) \tag{2.15}$$

Inserting (2.15) into (2.8) and taking (2.9) into account we obtain the expressions analogous to (2.13) and (2.14)

$$\alpha_i \frac{\partial C_i^{(k-1)}}{\partial \tau} + \frac{\partial C_i^{(k-1)}}{\partial \chi} [A(V)V_{C_i}, \psi^{(i)}] = [\bar{h}^{(k)}, \psi^{(i)}] \tag{2.16}$$

$$(i = l+1, \ldots, l+r, \quad k = 1, 2, \ldots, m; \quad i = 1, 2, \ldots, l, \quad k = 2, 3, \ldots, m)$$

$$\alpha_i \frac{\partial^* C_i^{(0)}}{\partial t} + \frac{\partial^* C_i^{(0)}}{\partial x} [A(V)V_{C_i}', \psi^{(i)}] = -[B(V), \psi^{(i)}] + [\bar{h}^{(i)}, \psi^{(i)}] \qquad (i = 1, 2, \ldots, l) \tag{2.17}$$

We note that the system (2.13), (2.14) is more suitable for use in analysis, as instead of $(l + r)\,m$ first order equations it contains $l + r$ equations of order not higher than the degree of approximation. Being nonlinear, Eqs. (2.16) and (2.17) cannot be reduced to the form (2.13) and (2.14), although the converse is always possible.

3. To provide the substantiation of the asymptotic method considered, we must prove that for the system (1.1) the difference between the exact solution and its mth approximation satisfying the system (1.12) is of the order μ^m on the interval in question. We shall prove this for the Cauchy problem, for the hyperbolic and symmetrical parabolic systems.

To make the notation more compact, we shall write the systems (1.1) and (1.12) in the form

$$u_t + A(u, x, t, \mu)u_x + B(u, x, t, \mu) = \mu f(u, u_x, u_t, x, t, \mu) \tag{3.1}$$

$$u_t^{(m)} + A(u^{(m)}, x, t, \mu)u_x^{(m)} + B(u^{(m)}, x, t, \mu) = \mu f(u^{(m)}, u_x^{(m)}, u_t^{(m)}, x, t, \mu) + \mu^{m+1} g^{(m)}(x, t, \mu) \tag{3.2}$$

Here the dependence on the slow variables is included in the dependence on x and t, and a single term is taken from the right-hand side of (1.1).

Theorem 1. Let the systems (3.1) and (3.2) satisfy the following requirements in the region $0 \leqslant t \leqslant \infty$, $0 \leqslant \mu \leqslant \mu_1$, $-\infty < u, u_x, u_t, x < \infty$:

1) $A(u, x, t, \mu)$, $B(u, x, t, \mu)$, $f(u, u_x, u_t, x, t, \mu)$ and $g^{(m)}(x, t, \mu)$ are continuous in μ, continuously differentiable in x and t and twice continuously differentiable in u, u_x and u_t.

2) Solutions $u(x, t, \mu)$ and $u^{(m)}(x, t, \mu)$ of the systems (3.1) and (3.2) exist, are continuous in μ and twice continuously differentiable in x and t, and satisfy the initial conditions $u(x, 0, \mu) = \varphi_1(x, \mu)$ and $u^{(m)}(x, 0, \mu) = \varphi_2(x, \mu)$, where φ_1 and φ_2 are functions belonging to the class C^2 and $|\varphi_1 - \varphi_2| < K\mu^{m+1}$, where K is a constant.

3) The systems are hyperbolic and all n characteristics passing through any point on the half-plane (x, t) in the opposite direction, intersect the abscissa.

Then for any numbers X_1, X_2 and T a constant M and a value μ_0 exist such that

$$|u^{(m)}(x, t, \mu) - u(x, t, \mu)| < M\mu^{m+1}$$
$$\text{for all} \quad 0 \leqslant t \leqslant T$$
$$X_1 \leqslant x \leqslant X_2, \ 0 \leqslant \mu \leqslant \mu_0$$

Note. If the function f is independent of u_x and u_t or depends on them linearly, then it is sufficient to require that the functions B, f and $g^{(m)}$ are continuuous in x and t and to limit oneself to showing the existence of solution and initial conditions belonging to the class C^1.

Proof. We shall assume that $u^{(m)}$ and u are known. Substituting them into (3.1) and (3.2) we obtain identities in x and t. Let us subtract (3.1) from (3.2)

$$\begin{aligned}(u^{(m)} - u)_t + A(u^{(m)})(u^{(m)} - u)_x + [A(u^{(m)}) - A(u)]u_x + B(u^{(m)}) - B(u) = \\ = \mu[f(u^{(m)}, u_x^{(m)}, u_t^{(m)}) - f(u, u_x, u_t)] + \mu^{m+1}g^{(m)}(x, t)\end{aligned} \tag{3.3}$$

Now set $u^{(m)} - u = w$. By the Hadamard lemma on finite increments we have

$$A(u^{(m)}) - A(u) = C(u^{(m)}, u)w = C(x, t, \mu)w$$
$$B(u^{(m)}) - B(u) = E(u^{(m)}, u)w = E(x, t, \mu)w$$
$$\begin{aligned}f(u^{(m)}, u_x^{(m)}, u_t^{(m)}) - f(u, u_x, u_t) = f_3(u^{(m)}, u, u_x^{(m)}, u_x, u_t^{(m)}, u_t)\,w + \\ + f_2(u^{(m)}\ u, u_x^{(m)}, u_x, u_t^{(m)}, u_t)\,w_x + f_1(u^{(m)}, u, u_x^{(m)}, u_x, u_t^{(m)}, u_t)\,w_t\end{aligned}$$

Functions C, E, f_1, f_2 and f_3 are continuously differentiable, and depend in the end on x, t and μ. Taking these relations into account we can write (3.3) in the form

$$[I - \mu f_1]w_t + [A - \mu f_2]w_x + [Cu_x + E - \mu f_3]w = \mu^{n+1}g^{(m)} \tag{3.4}$$

where I is a unit matrix. Since f_1 and f_2 are continuous, a value μ_0 exists such that for all $0 \leqslant \mu \leqslant \mu_0$ system (3.4) remains hyperbolic in a bounded closed region D which shall be defined later. In addition there exists a linear transformation $w = Hw_1$ with a continuously differentiable nonsingular matrix H which reduces (3.4) to the form

$$w_{1_t} + A_1(x, t, \mu)w_{1_x} + B_1(x, t, \mu)w_1 = \mu^{m+1}g_1(x, \mu, t) \tag{3.5}$$

where A_1 is a symmetric, nonsingular matrix [10] of class C^1.

Let us perform the substitution $w_1 = e^{\alpha t}z$. Then (3.5) yields the following expression for $z(x, t)$:

$$z_t + A_1 z_x + (B_1 + \alpha I)z = \mu^{m+1}e^{-\alpha t}g_1 \tag{3.6}$$

By virtue of the symmetry of the matrix A_1 the identity

$$(z,\ A_1z)_x = (z_x,\ A_1z) + (z,\ A_1z_x) + (z,\ A_{1x}z)$$

implies

$$2\,(z,\ A_1z_x) = (z,\ A_1z)_x - (z,\ A_{1_x}z) \tag{3.7}$$

Following Courant [10] we scalar multiply (3.6) by z and take (3.7) into account, to obtain

$$\begin{gathered} {}^1\!/_2\,(z,\ z)_t + {}^1\!/_2\,(z,\ A_1z)_x + (z,\ [B_1 + \alpha I - {}^1\!/_2 A_{1_x}]\,z) = \mu^{m+1}\,(z,\ g_2) \\ g_2\,(x,\ t) = e^{-\alpha t} g_1\,(x,\ t) \end{gathered} \tag{3.8}$$

Let us choose the value of the parameter α large enough for the matrix $B_2 = B_1 + \alpha I - {}^1\!/_2 A_{1x}$ to be positive definite in the region D; $0 \leqslant t \leqslant T$, $X_{\min} \leqslant x \leqslant X_{\max}$ $(0 \leqslant \mu \leqslant \mu_0)$, where $X_{\min}$ is the abscissa of the point of intersection with the x-axis of the characteristic passing through the point $(X_1,\ T)$ at the smallest angle of inclination relative to this axis and $X_{\max}$ is the abscissa of the point of intersection with the x-axis of the characteristic passing through the point $(X_2,\ T)$ at the greatest angle of inclination to the x-axis (see Fig. 1).

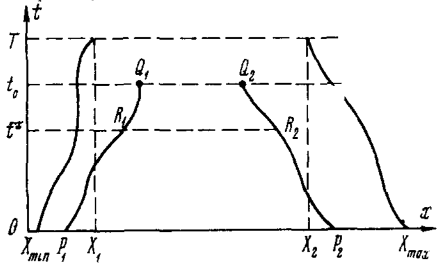

Fig. 1.

Let x_1 and x_2 be any numbers defined by $X_1 \leqslant x_1 < x_2 \leqslant X_2$, and t_0 any number $0 < t_0 \leqslant T$. We draw the extreme characteristics through the points $Q_1(x_1,\ t_0)$ and $Q_2\,(x_2, t_0)$ and denote by P_1 and P_2 the corresponding points of intersection of these characteristics with the abscissa (see Fig. 1).

Let us now integrate the identity (3.8) along the curvilinear trapeze $P_1P_2Q_2Q_1$. Applying the Green formula we obtain

$$\begin{aligned} &\frac{1}{2} \iint\limits_{P_1P_2Q_2Q_1} [(z,\ z)_t + (z,\ A_1z)_x]\,dxdt = \frac{1}{2} \int\limits_{P_1P_2Q_2Q_1} [(z,\ z)\,n_t + (z,\ A_1z)\,n_x]\,ds = \\ &= \frac{1}{2} \int\limits_{Q_1Q_2} z^2dx - \frac{1}{2} \int\limits_{P_1P_2} z^2dx + \frac{1}{2} \int\limits_{P_1Q_1+P_2Q_2} [(z,\ z)\,n_t + (z,\ A_1z)\,n_x]\,ds = \\ &= \frac{1}{2} \int\limits_{Q_1Q_2} z^2dx - \frac{1}{2} \int\limits_{P_1P_2} z^2dx + \frac{1}{2} \int\limits_{P_1Q_1+P_2Q_2} n_x \left(z, \left[A_1 + \frac{n_t}{n_x} I\right] z\right) ds \end{aligned} \tag{3.9}$$

where n_x and n_t denote the components of the unit normal.

Let us denote by λ_1 the largest and by λ_2 the smallest eigenvalue of the matrix A_1 On the characteristic P_1Q_1 we have $n_t\,/\,n_x = -\lambda_1$ and on P_2Q_2 we have $n_t\,/\,n_x = -\lambda_2$. For a symmetric matrix

$$\lambda_1 = \max \frac{(z,\ A_1z)}{(z,\ z)}, \qquad \lambda_2 = \min \frac{(z,\ A_2z)}{(z,\ z)}$$

Therefore $\lambda_1\,(z,\ z) \geqslant (z,\ A_1z)$ and $\lambda_2\,(z,\ z) \leqslant (z_1,\ A_1z)$ or $(z,\ [A_1 - \lambda_1 I]\ z) \leqslant 0$ and $(z,\ [A_1 - \lambda_2 I]\ z) \geqslant 0$. Since $n_x < 0$ on P_1Q_1 and $n_x > 0$ on P_2Q_2, we have

$$\frac{1}{2} \int\limits_{P_1Q_1+P_2Q_2} n_x \left(z, \left[A_1 + \frac{n_t}{n_x} I\right] z\right) ds \geqslant 0 \tag{3.10}$$

Moreover, by virtue of the initial conditions

$$\int\limits_{P_1P_2} z^2 dx < K_1^2 X \mu^{2(m+1)}, \qquad X = X_2 - X_1, \qquad K_1 = K \, \| H^{-1} \| \tag{3.11}$$

where $\| H^{-1} \|$ denotes the norm of the matrix H^{-1} The matrix B_2 is positive definite, therefore

$$\iint\limits_{P_1P_2Q_2Q_1} (z, B_2 z) \, dx \, dt \geqslant 0 \tag{3.12}$$

Taking into account (3.9) – (3.12) and integrating (3.8), we obtain the inequality

$$\frac{1}{2} \int\limits_{x_1}^{x_2} z^2 (x, t_0) \, dx < \mu^{m+1} \iint\limits_{P_1P_2Q_2Q_1} (z, g_2) \, dx \, dt + \frac{1}{2} K_1^2 X \mu^{2(m+1)} \tag{3.13}$$

Let us denote by $M_1 / 2$ the maximum value of the modulus of the vector function g_2 in the region D, i.e. $M_1 = 2 \max | e^{-\alpha t} g_2 (x, t, \mu) |$, $(x, t) \in D$ $(0 \leqslant \mu \leqslant \mu_0)$. Then

$$\int\limits_{x_1}^{x_2} z^2 (x, t_0) \, dx < M_1 \mu^{m+1} \int\limits_0^{t_0} dt \int\limits_{\varphi_1 (t)}^{\varphi_2 (t)} | z (x, t) | \, dx + K_1^2 X \mu^{2(m+1)} \tag{3.14}$$

where $x = \varphi_1 (t)$ is the equation of the characteristic P_1Q_1 and $x = \varphi_2 (t)$ of P_2Q_2. The integral

$$\int\limits_{\varphi_1 (t)}^{\varphi_2 (t)} | z (x, t) | \, dx$$

being a function of t, is of the same order of magnitude, namely μ^{m+1} for all values of t including $t = 0$.

Indeed, let it attain its maximum value at some t^* $(0 \leqslant t^* \leqslant t_0)$. Setting in (3.14) $t = t^*$, we obtain

$$\int\limits_{x_1 = \varphi_1 (t_0)}^{x_2 = \varphi_2 (t_0)} z^2 (x, t_0) \, dx < T M_1 \mu^{m+1} \int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t^*) | \, dx + K_1^2 X \mu^{2(m+1)}$$

By virtue of the Schwartz inequality we have

$$\left(\int\limits_{\varphi_1 (t_0)}^{\varphi_2 (t_0)} | z (x, t_0) | \, dx \right)^2 < T X M_1 \mu^{m+1} \int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t^*) | \, dx + K_1^2 X^2 \mu^{2(m+1)}$$

Setting $t_0 = t^*$, i.e. integrating (3.8) along the trapeze $P_1P_2R_2R_1$ where $R_1 (\varphi_1 (t^*), t^*)$ and $R_2 (\varphi_2 (t^*) \; t^*)$, we obtain

$$\left(\int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t) | \, dx \right)^2 < T X M_1 \mu^{m+1} \int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t^*) | \, dx + K_1^2 X^2 \mu^{2(m+1)} \tag{3.15}$$

from which follows

$$\int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t^*) | \, dx < M_2 \mu^{m+1}, \quad M_2 = \frac{1}{2} T X \left[M_1 + \left(M_1^2 + \frac{4 K_1^2}{T^2} \right)^{1/2} \right] \tag{3.16}$$

By the definition of t^* we have

$$\int\limits_{\varphi_1 (t_0)}^{\varphi_2 (t_0)} | z (x, t_0) | \, dx \leqslant \int\limits_{\varphi_1 (t^*)}^{\varphi_2 (t^*)} | z (x, t^*) | \, dx$$

or, with (3.16) taken into account

$$\int_{x_1}^{x_2} |z(x,\ t)|\,dx < M_2\mu^{m+1} \tag{3.17}$$

Having obtained the estimate for the integral and taking into account the boundedness of $z(x, t)$ and $z_x(x, t)$ in the region D we can estimate the integrand function. If the value of the integral is fixed, the maximum value of the integrand function belonging to class C^1 is

$$\int_{x_1}^{x_2} |z|\,dx = \frac{(\max|z|)^2}{\max|z|_x}$$

Consequently, in the general case we have

$$|z|^2 \leqslant \max|z|_x \int_{x_1}^{x_2} |z|\,dx \leqslant \max|z_x| \int_{x_1}^{x_2} |z|\,dx$$

or, in accordance with (3.17),

$$|z|^2 \leqslant \max|wz_x|\, M_2\mu^{m+1} \tag{3.18}$$

This yields the following estimate for $w_1(x, t)$

$$|w_1|^2 < \max|w_{1_x}|\, M_2 e^{\alpha T}\mu^{m+1} \tag{3.19}$$

It remains to estimate $w(x, t)$. Since $w_1 = H^{-1}w$, we have

$$\max|w_{1_x}| \leqslant \|H^{-1}\| \max|w_x| + \|H_x^{-1}\| \max|w|$$

Let us denote $\max|w_x| / \max|w| = S$. Taking into account (3.19) we obtain

$$|w|^2 \leqslant \|H\|^2 |w_1|^2 < \|H\|^2 (\|H^{-1}\| S + \|H_x^{-1}\|) M_2 e^{\alpha T} \max|w|$$

$$|u^{(m)} - u| < M\mu^{m+1}, \quad M = TXe^{\alpha T} \|H\|^2 (\|H^{-1}\| S + \|H_x^{-1}\|)\ \frac{1}{2}\left(M_1 + \sqrt{M_1^2 + \frac{4K_1^2}{T^2}}\right)$$

which completes the proof of the theorem.

The parameter α appearing in the proof characterizes the degree of instability of the homogeneous system $N(u) = 0$. The asymptotic method can be applied to systems of arbitrary degree of instability. However, when the nearly periodic solutions of (3.1) are considered, α must be of the same order of magnitude as μ, in which case $e^{\alpha T}$ is of the order of unity when $T < 1/\mu$. Then, with H, S and g_2 having the order of unity on any finite intervals of x and t, the difference $u^{(m)} - u$ is of order μ^m on the interval $XT \sim 1/\mu$ (e.g. $X \sim 1/\sqrt{\mu}$, $T \sim 1/\sqrt{\mu}$).

From the proof given above we can infer that an analogous theorem holds for symmetrical parabolic systems which have at least two characteristics at each point of the half plane x, t On moving in the opposite direction all characteristics intersect the abscissa axis and the right-hand side of (3.1) is independent of u_x and u_t. In this case we obtain in place of (3.4) the following symmetrical system

$$w_t + Aw_x + [Cu_x + E - \mu f_3]w = \mu^{m+1} g^{(m)}$$

The substitution $w = Hw_1$ ceases to be necessary and all the arguments that follow, remain in force.

T h e o r e m 2. Let the system

$$u_t + A\ (u,\ x,\ t,\ \mu)\ u_x + B\ (u,\ x,\ t,\ \mu) = 0 \tag{3.20}$$

satisfy, in the region $-\infty < u,\ x, < \infty,\ 0 \leqslant t,\ \mu < \infty$, the following conditions:

1) A is a symmetric matrix continuously differentiable in all its arguments.

2) B is continuous in x and t, and continuously differentiable in u and μ.

3) At least two characteristics pass through each point of the half plane x, t and, on moving in the opposite direction, all characteristics intersect the x-axis.

4) $u(x, 0, \mu) = \varphi(x, \mu) \in C^1$.

Then $u(x, t, \mu)$ is a continuously differentiable function for all $0 \leqslant \mu < \infty$

Proof. Let $u(x, t, \mu)$ and $u(x, t, \mu + \Delta\mu)$ be known solutions of the system (3.20). Subtracting the corresponding identities we obtain

$$u_t(\mu + \Delta\mu) - u_t(\mu) + A[u(\mu + \Delta\mu), \mu + \Delta\mu][u_x(\mu + \Delta\mu) - u_x(\mu)] +$$
$$+ \{A[u(\mu + \Delta\mu), \mu + \Delta\mu] - A[u(\mu), \mu]\} u_x(\mu) + B[u(\mu + \Delta\mu), \mu + \Delta\mu] -$$
$$- B[u(\mu), \mu + \Delta\mu] + B[u(\mu), \mu + \Delta\mu] - B[u(\mu), \mu] = 0$$

(the arguments x and t are omitted here). Applying the Hadamard lemma to the corresponding differences we find

$$\Delta u_t + A\Delta u_x + C(\mu, \Delta\mu) u_x \Delta u + D(\mu, \Delta\mu) u_x \Delta\mu + E(\mu, \Delta\mu)\Delta u + F(\mu, \Delta\mu)\Delta\mu = 0$$

where C, D, E and F are continuous functions of x, t, μ and $\Delta\mu$. Treating $\Delta u = u(\mu + \Delta\mu) - u(\mu)$ as an unknown function, we obtain for it the following linear system:

$$\Delta u_t + A\Delta u_x + B^\circ \Delta u = \Delta\mu f^\circ, \quad B^\circ = Cu_x + E, \quad f^\circ = -Du_x - F \tag{3.21}$$

which satisfies the conditions of Theorem 1. A constant C° therefore exists such that $|\Delta u| < C^\circ |\Delta\mu|$, i.e. $u(x, t, \mu)$ is continuous in μ.

Dividing (3.21) by $\Delta\mu$ we obtain

$$\left(\frac{\Delta u}{\Delta\mu}\right)_t + A\left(\frac{\Delta u}{\Delta\mu}\right)_x + B^\circ \frac{\Delta u}{\Delta\mu} = f^\circ \tag{3.22}$$

which we shall now consider together with the linear system

$$z_t + Az_x + B_z = f^\circ \tag{3.23}$$

The solution $z(x, t, \mu, \Delta\mu)$ of the latter is a continuous function of the parameter $\Delta\mu$, including $\Delta\mu = 0$. Considering this solution with the initial conditions of (3.22) we obtain, by virtue of its uniqueness, the solution $z(x, t, \mu, \Delta\mu) = \Delta u / \Delta\mu$, and the following limit exists

$$\lim_{\Delta\mu \to 0} z(x, t, \mu, \Delta\mu) = z(x, t, \mu, 0) = \partial u / \partial\mu$$

To prove that u_μ is continuous we set in (3.22)

$$\Delta\mu \to 0, \quad \partial u_\mu / \partial t + A \partial u_\mu / \partial x + [A_u u_x + B_\mu] u_\mu = -B_\mu - A_\mu u_x$$

This system, linear in u_μ, satisfies the conditions which ensure that it solutions are continuous in μ. Thus $u(x, t, \mu) \in C^1$.

We shall show that if $F^{(i)}$ is chosen in such a manner that a bounded solution of (1.10) for $w^{(i)}$ exists, then under the restrictions ensuing from Theorem 2, $u^{(m)}$ exists defined by (1.5) and (1.7) and satisfying (1.12). Since $C_1, \ldots, C_{l+r}$ can be assumed to be infinitely differentiable functions of χ and τ, conditions of the Theorem 2 hold for the coefficients of the equation $Lw^{(1)} = h^{(1)} - F^{(1)}$ and $w^{(1)}(x, t, \chi, \tau) \in C^1$. Consequently $h^{(2)}[w_\chi^{(1)}, w_\tau^{(1)}] = h^{(2)}(x, t, \chi, \tau) \in C^1$ and $w^{(2)} \in C^1$. In general, since $w^{(i)}$ are determined with the help of recurrent relations, then $w^{(i-1)} \in C^1$ implies that $w^{(i)} \in C^1$ $(i = 2, 3, \ldots, m)$ Taking into account the composition of $g^{(m)}(x, t, \chi, \tau)$ we find that $g^{(m)} \in C^1$, i.e. conditions of the existence theorem hold for (1.12).

Applying Theorems 1 and 2 to the systems (1.1), (1.12), we are in position to state the basic result in the form of the following theorem.

Theorem 3. Let the system (1.1) be hyperbolic in the region $0 \leqslant t, \tau < \infty$ $-\infty < u, u_x, u_t, x, \chi < \infty$ and satisfy the following requirements within this region:

(1) A, B and f_k are differentiable $m+1$ times in u, u_x and u_t, and continuously differentiable in x t, χ and τ.

(2) A solution of (1.1) exists, twice continuously differentiable in x and t, and continuous in μ.

(3) Through each point of the (x, t)-plane pass n characteristics and when moving in the opposite direction, all these characteristics intersect the x-axis.

(4) Let $u^{(m)}$ be defined by the formulas (1.5) and (1.7) where $w^{(i)}$ are solutions of (1.10) and $F^{(i)}$ are chosen so, that solutions of (1.10) restricted in the sense of (1.11) exist when the initial conditions follow from

$$|u^{(i)}(x, 0, \mu) - u(x, 0, \mu)| < K^{(i)}\mu^{i+1} \qquad (i = 1, 2, \ldots, m; \quad K^{(i)} = \mathrm{const})$$

Then μ_0 exists such that for all $0 \leqslant \mu \leqslant \mu_0$ and x, t taken from the interval $XT \sim 1/\mu$ $\qquad |u^{(m)}(x, t, \mu) - u(x, t, \mu)| < M^{(m)}\mu^m$

The magnitude of the constant $M^{(m)}$ depends on $\max |g^{(m)}(x, t, \chi, \tau)|$ and estimation of the latter represents a problem interesting in itself.

Analogous assertion is valid for the parabolic system

$$u_t + A(u, x, t, \chi, \tau) u_x + B(u, x, t, \chi, \tau) = \sum_i \mu^i f_i(u, x, t, \chi, \tau)$$

satisfying the following conditions in the region $-\infty < u, x, \chi < \infty; 0 \leqslant \tau, t < < \infty$:

(1) A, B and f_i are differentiable $m+1$ times in u and continuously differentiable in x, t, χ and τ.

(2) A solution $u(x, t, \mu) \in C^1$ of this system exists.

(3) At least two characteristics pass through each point of the half plane (x, t) and, when moving in the opposite direction, all these characteristics intersect the x-axis.

The authors express their gratitude to A. V. Gaponov for continued interest in this work and to V. N. Gol'dberg for useful discussions.

BIBLIOGRAPHY

1. Migdal A. B. and Krainov V. P., Approximate methods in Quantum Mechanics. M., "Nauka", 1966.
2. Moiseev N. N., Asymptotic Methods in Nonlinear Mechanics. M., "Nauka", 1969.
3. Rytov S. M., Modulated oscillations and waves. Tr. FIAN SSSR, Vol. 2, 1940.
4. Ostrovskii L. A. and Pelinovskii E. N., The method of averaging for nonsinusoidal waves. Dokl. Akad. Nauk SSSR, Vol. 195, №4 1970.
5. Rabinovich M. I., On an asymptotic method in the theory of nonlinear oscillations of distributed systems. Dokl. Akad. Nauk SSSR, Vol. 191, №6, 1970.
6. Gaponov A. V., Ostrovskii L. A. and Rabinovich M. I., Asymptotic methods in the nonlinear theory of oscillations of distributed systems. Proc.

of the fifth intern. conf. on nonlinear oscillations., Vol. 1, "Naukova Dumka" Kiev, 1970.

7. Bogoliubov N. N. and Mitropol'skii Iu. A., Asymptotic Methods in the Theory of Nonlinear Oscillations. M., Fizmatgiz, 1963.

8. Bers L., John F. and Schechter M., Partial Differential Equations., N.Y., Interscience, 1964.

9. Gaponov A. V., Ostrovskii L. A. and Rabinovich M. I., One-dimensional waves in nonlinear systems with dispersion. Izv. vuzov., Radiofizika, Vol. 13, №2, 1970.

10. Courant R., Partial Differential Equations (Russian Transl.) M., "Mir", 1964.

Translated by L. K.

UDC 534.015

LIAPUNOV SYSTEMS WITH DAMPING

PMM Vol. 36, №2, 1972, pp. 344-348

V. M. STARZHINSKII
(Moscow)

(Received February 25, 1970)

A nonlinear, autonomous system of order ($2k + 2$) is perturbed by application of damping which is analytic and sufficiently small in norm. The system we consider resembles a Liapunov system [1], in a different sense however to that given in [2]. The perturbed system is transformed in such a manner that the unperturbed system transforms into a quasilinear, nonautonomous system of order $2k$ [3]. If the general solution to the unperturbed system is known, then the process of integration of the system of variational equations can be reduced, according to Poincare [4], to quadratures and this is illustrated with the example of a plane spring pendulum.

1. Transformation of the equations of motion. Consider a class of Liapunov systems (see [1], Sect. 33) with damping, described by the following system of equations:

$$d^2u/d\tau^2 + u - U(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) = -2\varepsilon F_0(\dot{u}, \dot{v}_1, \ldots, \dot{v}_k) \qquad (1.1)$$

$$d^2v_\varkappa/d\tau^2 + a_{\varkappa 1}v_1 + \ldots + a_{\varkappa k}v_k - V_\varkappa(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) =$$

$$= -2\varepsilon F_\varkappa(\dot{u}, \dot{v}_1, \ldots, \dot{v}_k) \quad (\varepsilon > 0, \varkappa = 1, \ldots, k)$$

Here a dot denotes a derivative with respect to τ ; $a_{j\varkappa} = a_{\varkappa j}$ $(\varkappa, j = 1,\ldots,k)$ are real constants; U, $V_1,\ldots,V_k$, F_0, $F_1,\ldots,F_k$ are real analytic functions; the expansions for F_0, F_1, $\ldots, F_k$ begin with the terms of at least first order and those for U, $V_1,\ldots,V_k$ with terms of at least second order. We shall assume that the unperturbed system (1.1), i.e. (1.1) in which $\varepsilon = 0$, admits a first integral which must be an analytic function of the variables $u, \dot{u}, v_1,\ldots, v_k, \dot{v}_1,\ldots, \dot{v}_k$ and have the form [1]

$$H = \dot{u}^2 + u^2 + W(v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) +$$

$$+ S_3(u, \dot{u}, v_1, \ldots, v_k, \dot{v}_1, \ldots, \dot{v}_k) = \mu^2 \quad (\mu > 0) \qquad (1.2)$$

where W is a quadratic form and S_3 is a set of terms of order not lower than the third.